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Fractal Random Walks

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We consider a class of random walks (on lattices and in continuous spaces) having infinite mean-squared displacement per step. The probability distribution functions considered generate fractal self-similar trajectories. The characteristic functions (structure functions) of the walks are nonanalytic functions and satisfy scaling equations.

KEY WORDS: random walks; stochastic processes; fractals; scaling; stable distributions; nondifferentiable functions.

1. INTRODUCTION

Many of the complex systems of interest today are without an immediately apparent scale of length. Pattern formation may exhibit a strange structure of clusters within clusters or waves within waves with no limit in either direction. The purpose of this paper is to present some random walk models of sufficient flexibility to exhibit interesting hierarchical structures. These random walk processes involve unusual probability distributions for the displacement per step. For certain parameter regimes, the walks have infinite spatial moments, leading to nonstandard continuum limits and unusual statistics. Perhaps more interestingly, these walks have the following additional properties: (i) they provide a simple realization in stochastic processes of the "fractals" of Mandelbrot$^{(1)}$; (ii) the characteristic functions (structure functions) have highly nonanalytic behavior at all points; (iii)

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they lead to an analog, in a probabilistic context, of real-space renormalization group transformations.[2]

In Section 2 we summarize the basic properties of (Markovian) random walks, with particular emphasis on the qualitative differences between walks with a finite mean-squared displacement per step, which are ultimately diffusive in character, and walks with infinite mean-squared displacement per step. The latter case includes the discrete-time version of the "Lévy flight" discussed by Mandelbrot[1] (as an example of a stochastic process with a "fractal" trajectory), but the relation to fractals is made more explicit by particular examples presented in Sections 3 and 4. The one-dimensional example of Section 3 ("Weierstrass random walk") has been discussed briefly elsewhere,[3] and its structure function is the celebrated nondifferentiable function of Weierstrass.[4] Its higher-dimensional analog is given in Section 4. Section 5 discusses fractals and Section 6 gives a generating function formalism for continuous-time processes.

2. DIFFUSIVE AND NONDIFFUSIVE RANDOM WALKS

For a random walk in discrete time, taking place in a space of dimension s, we denote by $P_n(\mathbf{x})$ the probability density function (p.d.f.) for the position $\mathbf{X}_n$ of the walker after n steps and we let $p(\mathbf{x})$ denote the p.d.f. for the displacement of the walker at each step (i.e., the p.d.f. of the random variables $\mathbf{X}_{n+1} - \mathbf{X}_n$, which are taken as independent and identically distributed). Since

$$P_{n+1}(\mathbf{x}) = \int p(\mathbf{x}-\mathbf{y}) P_n(\mathbf{y})\, d^s\mathbf{y} \tag{1}$$

the solution for $P_n(\mathbf{x})$, given that the walker starts from the origin [i.e., $P_0(\mathbf{x}) = \delta(\mathbf{x})$], is easily found by Fourier transforming:

$$P_n(\mathbf{x}) = \frac{1}{(2\pi)^s} \int \exp(-i\mathbf{q}\cdot\mathbf{x}) \tilde{P}_n(\mathbf{q})\, d^s\mathbf{q} \tag{2}$$

where

$$\tilde{P}_n(\mathbf{q}) = \{\tilde{p}(\mathbf{q})\}^n \tag{3}$$

and

$$\tilde{p}(\mathbf{q}) = \int \exp(i\mathbf{q}\cdot\mathbf{x}) p(\mathbf{x})\, d^s\mathbf{x} \tag{4}$$

The qualitative features of the walk are determined by the behavior of $p(\mathbf{x})$ as $|\mathbf{x}| \to \infty$, or equivalently by the behavior of $\tilde{p}(\mathbf{q})$ as $|\mathbf{q}| \to 0$. If $p(\mathbf{x})$ decays sufficiently rapidly as $|\mathbf{x}| \to \infty$ to ensure the existence of the first two cumulants

$$\langle \mathbf{x} \rangle = \int \mathbf{x} p(\mathbf{x})\, d^s\mathbf{x}, \qquad \sigma^2 = \int \mathbf{x}^2 p(\mathbf{x})\, d^s\mathbf{x} - \langle \mathbf{x} \rangle^2 \tag{5}$$

we then have[5]

$$\tilde{p}(\mathbf{q}) = 1 + i\mathbf{q}\cdot\langle\mathbf{x}\rangle - \tfrac{1}{2}\sum_{i=1}^{s} q_i^2\langle\mathbf{x}_i^2\rangle + o(|\mathbf{q}|^2) \tag{6}$$

It has been assumed here that

$$\langle x_1^2\rangle = \langle x_2^2\rangle = \cdots = \langle x_s^2\rangle \tag{7}$$

and

$$\langle x_i x_j\rangle = \langle x_i^2\rangle\delta_{ij} \tag{8}$$

As the number of steps becomes larger a continuous-time can be introduced and a diffusion constant D and velocity V can be defined in the standard manner. Gaussian behavior, in accordance with the central limit theorem[6] for $P(\mathbf{x}, t)$

$$P(\mathbf{x}, t) \sim \{4\pi D t\}^{-s/2}\exp\{-|\mathbf{x} - \mathbf{v}t|^2/(4Dt)\} \tag{9}$$

with the right-hand side being the solution, for the initial condition $P(\mathbf{x}, 0) = \delta(\mathbf{x})$, of the Fokker–Planck (or Smoluchowski) equation[7]

$$\frac{\partial P}{\partial t} = -\mathbf{v}\cdot\nabla P + D\nabla^2 P \tag{10}$$

When $\langle\mathbf{x}\rangle = \mathbf{0}$ we obtain the classical diffusion equation, while if $\langle\mathbf{x}\rangle \neq \mathbf{0}$ we have a diffusing packet of probability with center-of-mass velocity $\mathbf{v}$.

In the present paper we shall be considering walks for which $\langle\mathbf{x}^2\rangle = \infty$, so that the long-time behavior is not diffusive, apart from borderline cases in which the integral (5) defining σ^2 diverges slowly enough that (9) holds, with a nonstandard scaling of length and time replacing the usual diffusion constant $D = \sigma^2/(2\tau)$. [We shall encounter such borderline cases in specific examples considered in the paper. They illustrate the general theorem[8] that for a one-dimensional walk with $p(x)$ not concentrated at one point, we obtain a Gaussian distribution in the large n limit if and only if

$$\int_{-y}^{y} x^2 p(x)\,dx \sim L(y) \qquad \text{as} \quad y \to \infty \tag{11}$$

with $L(y)$ slowly varying, i.e., $L(\lambda y)/L(y) \to 1$ as $y \to \infty$, for all $\lambda > 0$.]

If a nondiffusive process is to be a stationary stochastic process with translational invariance it must satisfy the (Bachelier–Smoluchowski–Chapman–Kolmogorov) chain equation[7]

$$P(\mathbf{x}, t) = \int P(\mathbf{x} - \mathbf{y}, t')P(\mathbf{y}, t - t')\,d^s\mathbf{y} \tag{12}$$

and we are led to the Lévy (stable) distributions, which are characterized in one spatial dimension by

$$\tilde{P}(q,t) = \exp\{i\gamma q t - ct|q|^{\mu}[1 + i\nu\omega(q,\mu)\operatorname{sign} q]\} \tag{13}$$

where μ, ν, γ, and c are real constants, with $0 < \mu \leqslant 2$, $-1 \leqslant \nu \leqslant 1$, $c \geqslant 0$ and

$$\omega(q,\mu) = \begin{pmatrix} \tan(\frac{1}{2}\pi\mu) & \text{if} \quad \mu \neq 1 \\ (2/\pi)\ln|q| & \text{if} \quad \mu = 1 \end{pmatrix} \tag{14}$$

For $\mu = 2$, $P(x,t)$ is Gaussian and all its spatial moments (including both the mean and variance) are finite. However, if $\mu < 2$, the variance is infinite, since in this case

$$P(x,t) = O(|x|^{-1-\mu}) \qquad \text{as} \quad |x| \to \infty \tag{15}$$

Moreover the mean displacement is not well defined when $\mu \leqslant 1$, even for the symmetric or unbiased case $[P(x,t) = P(-x,t)]$, since $\int xP(x,t)\,dx$ is not absolutely convergent.

It is possible to generalize the theory of stable distributions to more than one spatial dimension[9] using multidimensional Fourier transforms. For the present paper we require only the result that an *isotropic stable distribution* of order μ $(0 < \mu \leqslant 2)$ in s spatial dimensions has the multidimensional Fourier transform representation

$$\tilde{P}(\mathbf{q},t) = \int \exp(i\mathbf{q}\cdot\mathbf{x})P(\mathbf{x},t)\,d^s\mathbf{x} = \exp(-ct|\mathbf{q}|^{\mu}) \tag{16}$$

Using the Fourier inversion theorem for radial functions,[10] the p.d.f. in (16) may be expressed in terms of a single integral involving the usual Bessel function:

$$P(\mathbf{x},t) = (2\pi)^{-s/2}|\mathbf{x}|^{1-s/2}\int_0^{\infty} q^{(1/2)s}J_{s/2-1}(q|\mathbf{x}|)\exp(-ctq^{\mu})\,dq \tag{17}$$

Only the cases $\mu = 2$ and $\mu = 1$ lead to simple expressions for $P(x,t)$ in terms of elementary functions, the former being the s-dimensional Gaussian distribution and the latter the s-dimensional generalization of the Cauchy distribution,

$$P(x,t) = \frac{\Gamma(\frac{1}{2}s + \frac{1}{2})ct}{\pi^{s/2+1/2}(|x|^2 + c^2t^2)^{s/2+1/2}} \tag{18}$$

3. THE WEIERSTRASS RANDOM WALK

We consider a symmetric random walk on the one-dimensional continuum for which the p.d.f. for a displacement x at any step is

$$p(x) = \frac{a-1}{2a}\sum_{n=0}^{\infty} a^{-n}\{\delta(x - \Delta b^n) + \delta(x + \Delta b^n)\} \tag{19}$$

with a, b, and Δ constants, $a > 1$, $b > 1$, and $\Delta > 0$. The constant Δ has the dimensions of length and is introduced to facilitate passage to a continuum limit, while a and b are dimensionless. If b is integral, then the walk is confined to a lattice of spacing Δ. The p.d.f. (19) has the property that a step of length Δb^n is a times more likely than the next longest step (Δb^{n+1}). It follows, roughly speaking, that the walker will make on the average about a steps of a given order of magnitude (and many smaller steps), forming a cluster, before moving an order of magnitude further away and beginning a new cluster. If the walk is terminated after a modest number of steps, then whatever the values of a and b, a clustered pattern will be seen. This does not guarantee, however, that distinguishable clusters will persist as the number of steps taken becomes large, and we return to this point in Section 5.

The mean-squared displacement per step is

$$\langle x^2 \rangle = \frac{a-1}{a} \sum_{n=0}^{\infty} \{b^2/a\}^n \Delta^2 \tag{20}$$

so that when $b^2 < a$, $\langle x^2 \rangle$ is finite and the walk has diffusive behavior after a large number of steps. The case $b^2 \geqslant a$ requires a separate analysis to determine the behavior of $\tilde{p}(q)$ in the neighborhood of $q = 0$. We write

$$\tilde{p}(q) = \lambda(\Delta |q|) \tag{21}$$

where

$$\lambda(k) = \frac{a-1}{a} \sum_{n=0}^{\infty} a^{-n} \cos(b^n k) \tag{22}$$

[Since $p(x)$ is discrete, the Fourier transform reduces to a Fourier series. When b is integral, $\lambda(k)$ is the *structure function* in the terminology of lattice walk theory.[11,12]] The series (22) was first considered[4] by K. Weierstrass (in the latter half of the nineteenth century) as an example of a function which is everywhere continuous, but nowhere differentiable with respect to k (in certain parameter regimes). Hardy[13] has established that if $b \geqslant a$, $\lambda(k)$ has a *finite* derivative at no value of k [and it is known that under more restrictive conditions on a and b, $\lambda(k)$ even fails to have a well-defined infinite derivative, i.e., a vertical tangent, at any value of k]. The qualitative form of $\lambda(k)$ may be inferred directly from the functional equation

$$\lambda(k) = a^{-1} \lambda(bk) + \frac{a-1}{a} \cos k \tag{23}$$

which the series (22) clearly satisfies. If we define

$$\lambda_h(k) = \frac{a-1}{a} \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n}}{(2n)!(1 - a^{-1}b^{2n})} \tag{24}$$

we are easily able to verify by direct substitution that $\lambda_h(k)$ is a holomorphic function of k which satisfies (23). Hence

$$\lambda(k) = \lambda_s(k) + \lambda_h(k) \tag{25}$$

where $\lambda_s(k)$ contains all the singular behavior of $\lambda(k)$, and

$$\lambda_s(k) = a^{-1}\lambda_s(bk) \tag{26}$$

If we write

$$\lambda_s(k) = k^{\mu} Q_{\mu}(k) \tag{27}$$

then the choice

$$\mu = \ln a / \ln b \tag{28}$$

leads to the functional equation

$$Q_{\mu}(k) = Q_{\mu}(bk) \tag{29}$$

so that

$$\lambda(k) = \frac{a-1}{a} \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n}}{(2n)!(1 - a^{-1}b^{2n})} + k^{\mu} Q_{\mu}(k) \tag{30}$$

where Q_{μ} is a periodic function of $\ln k$ with period $\ln b$. (This argument is similar to one used in connection with the real-space renormalization group analysis of the free energy of an Ising lattice,[2,14] but the basic idea appears in a paper of Hardy,[15] who acknowledges its suggestion to him by a Mr. J. H. Maclagan Wedderburn.)

A certain amount of analysis is necessary to exhibit the explicit form of $Q_{\mu}(k)$ and we defer the details to the Appendix. The basic idea, however, is simple enough and is used elsewhere in the paper. Replacing the cosine in (22) by a Mellin integral representation, we are able to pass from the series (22) to the contour integral

$$\lambda(k) - 1 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(a-1)\Gamma(p)\cos(\frac{1}{2}\pi p)k^{-p}}{a(1 - a^{-1}b^{-p})}\, dp \tag{31}$$

$(-\mu < c = \operatorname{Re} p < 0)$. Translation of the contour towards $\operatorname{Re} p = -\infty$ and use of the residue theorem returns a new series expansion for $\lambda(k)$, of the form (30), with

$$Q_{\mu}(k) = \frac{a-1}{a \ln b} \sum_{n=-\infty}^{\infty} \Gamma\left(-\mu + \frac{2n\pi i}{\ln b}\right) \cos\left(\frac{\pi}{2}\left[-\mu + \frac{2n\pi i}{\ln b}\right]\right) \times \exp\left(-\frac{2n\pi i \ln k}{\ln b}\right) \tag{32}$$

provided that $1/2 < \mu < 2$. When $0 < \mu \leqslant 1/2$, the series for Q_μ must be summed using an appropriate convergence factor, while when $\mu = 2$, i.e., $b^2 = a$, we find from the contour integral argument that

$$\lambda(k) = 1 + \frac{a-1}{a} \sum_{n=2}^{\infty} \frac{(-1)^n k^{2n}}{(2n)!\{1 - a^{n-1}\}}$$
$$- \frac{(a-1)k^2}{2a \ln a} \left\{ \ln\left(\frac{1}{k}\right) + \frac{1}{4}\ln a + \frac{3}{2} - \gamma \right\} + k^2 \hat{Q}_2(k) \tag{33}$$

where $\hat{Q}_\mu(k)$ is defined by the same series as $Q_\mu(k)$, except for the omission of the $n = 0$ term, and $\gamma \simeq 0.5772$ denotes Euler's constant.

It is possible to exhibit a formal continuum limit for the Weierstrass walk by allowing the length scale Δ and the time τ between steps to approach zero together in a suitable fashion. From (1),

$$\frac{1}{\tau}\{P_{n+1}(x) - P_n(x)\} = \int_{-\infty}^{\infty} \frac{1}{\tau}\{p(x - x') - \delta(x - x')\} P_n(x')\,dx' \tag{34}$$

so that as $\Delta, \tau \to 0$,

$$\frac{\partial}{\partial t} P(x,t) = \int_{-\infty}^{\infty} \lim_{\Delta,\tau\to 0} \left\{ \frac{1}{\tau}\left[p(x-x') - \delta(x-x')\right]\right\} P(x',t)\,dx' \tag{35}$$

or, in Fourier space,

$$\frac{\partial}{\partial t}\tilde{P}(q,t) = \lim_{\Delta,\tau\to 0}\left\{\frac{1}{\tau}\left[\tilde{p}(q) - 1\right]\right\}\tilde{P}(q,t) \tag{36}$$

When $\mu > 2$ (so that $\langle x^2 \rangle < \infty$) it follows from (6) that (36) reduces to the Fourier transform of the diffusion equation. However (36) also remains useful for $0 < \mu \leqslant 2$. From (30) and (32) it is evident that for $\mu < 2$ the joint limit

$$\begin{pmatrix} a = 1 + \alpha\Delta = o(\Delta), & b = 1 + \beta\Delta + o(\Delta) \\ \Delta^\mu/\tau \sim \text{const} & 0 < \alpha < 2\beta \end{pmatrix} \tag{37}$$

yields[3]

$$\frac{\partial}{\partial t}\tilde{P}(q,t) = -D_1 |q|^{\alpha/\beta} \tilde{P}(q,t) \tag{38}$$

with D_1 a constant; i.e., $P(x,t)$ has a symmetric Lévy distribution of order $\alpha/\beta < 2$. In the borderline case $\mu = 2$ a joint limit $\Delta, \tau \to 0$ based on a power-law scaling between length and time does not suffice, but if we take

$$\begin{pmatrix} a = 1 + \alpha\Delta + o(\Delta), & b = 1 + 2\alpha\Delta + o(\Delta) \\ \Delta^2 \ln(1/\Delta)/\tau = \text{const} & \end{pmatrix} \tag{39}$$

we obtain the diffusion equation (illustrating the parenthetical remark in Section 1 concerning walks with $\langle x^2 \rangle$ divergent, but not too rapidly divergent).

4. THE FRACTAL RAYLEIGH–PEARSON WALK

The problem of a random walk in the plane for which the step length is fixed, but the possible step directions are isotropically distributed, was first solved (in a different context) by Lord Rayleigh,[16] but is usually named after K. Pearson,[17] who first posed the problem in the present context. A generalization of Pearson's walk to s spatial dimensions and variable step length is obtained by writing

$$p(\mathbf{x}) = \{A_s|\mathbf{x}|^{s-1}\}^{-1} p_1(|\mathbf{x}|) \tag{40}$$

with $p_1(x)$ the p.d.f. for a step of length x $(0 \leqslant x < \infty)$ and $A_s = 2\pi^{s/2}/\Gamma(\frac{1}{2}s)$ the surface area of the unit hypersphere $|\mathbf{x}| = 1$. Using the theory of Fourier transforms for radial functions[10] it may be shown that

$$\tilde{p}(\mathbf{q}) = \Gamma(\tfrac{1}{2}s)\int_0^\infty (\tfrac{1}{2}|\mathbf{q}|\xi)^{1-s/2} J_{s/2-1}(|\mathbf{q}|\xi) p_1(\xi)\, d\xi \tag{41}$$

To generate a clustered trajectory we choose (by analogy with the one-dimensional Weierstrass walk)

$$p_1(|\mathbf{x}|) = \frac{a-1}{a}\sum_{n=0}^{\infty} a^{-n}\delta(|\mathbf{x}| - \Delta b^n) \tag{42}$$

and find that

$$\tilde{p}(\mathbf{q}) = \frac{a-1}{a}\sum_{n=0}^{\infty} a^{-n}\Gamma(\tfrac{1}{2}s)(\tfrac{1}{2}|\mathbf{q}|\Delta b^n)^{1-s/2} J_{s/2-1}(|\mathbf{q}|\Delta b^n) \tag{43}$$

The general form of $\tilde{p}(\mathbf{q})$ may be obtained from a functional equation, as in Section 3,

$$\tilde{p}(\mathbf{q}) = a^{-1}\tilde{p}(b\mathbf{q}) + \frac{a-1}{a}\Gamma(\tfrac{1}{2}s)(\tfrac{1}{2}|\mathbf{q}|\Delta)^{1-s/2} J_{s/2-1}(|\mathbf{q}|\Delta) \tag{44}$$

and a detailed representation of $\tilde{p}(\mathbf{q})$ is derived in the Appendix. With $k = |\mathbf{q}|\Delta$ and $\mu = \ln a/\ln b$ as before, we find

$$\begin{aligned}\tilde{p}(\mathbf{q}) = \frac{a-1}{a}&\sum_{n=0}^{\infty}\frac{\Gamma(\frac{1}{2}s)(-1)^n(\frac{1}{2}k)^{2n}}{n!\,\Gamma(n+\frac{1}{2}s)(1-a^{-1}b^{2n})}\\ &+\frac{a-1}{2a\ln b}(\tfrac{1}{2}k)^{\mu}\Gamma(\tfrac{1}{2}s)\\ &\times\sum_{m=-\infty}^{\infty}\frac{\Gamma(-\frac{1}{2}\mu + m\pi i/\ln b)}{\Gamma(\frac{1}{2}s+\frac{1}{2}\mu - m\pi i/\ln b)}\exp\left[-\frac{2m\pi i\ln(\frac{1}{2}k)}{\ln b}\right]\end{aligned} \tag{45}$$

for $0 < \mu < 2$, while if $\mu = 2$ the $n = 1$ term in the first sum and the $m = 0$

term in the second sum are replaced by

$$-\frac{(a-1)\Gamma(\frac{1}{2}s)(\frac{1}{2}k)^2}{a\ln a\Gamma(1+\frac{1}{2}s)}\{\ln(2/k)+\tfrac{1}{4}\ln a+\tfrac{1}{2}\psi(1+\tfrac{1}{2}s)+\tfrac{1}{2}\psi(2)\} \quad (46)$$

Here[18] $\psi(z)=(d/dz)\ln\Gamma(z)$ and $\psi(2)=1-\gamma$, $\psi(\frac{5}{2})=-\gamma-2\ln 2+8/3$, $\psi(z+1)=\psi(z)+1/z$. The differentiability of the series (43) may be analysed using the identity[18]

$$\frac{d}{dz}\{z^{-\nu}J_\nu(z)\} = -z^{-\nu}J_{\nu+1}(z) \quad (47)$$

and a well-known theorem on term-by-term differentiation of series.[19] If $\mu>\frac{1}{2}(3-s)$, the structure function of the walk is differentiable with respect to $k=\Delta|\mathbf{q}|$ for all $k>0$. The condition $\mu>\frac{1}{2}(3-s)$ reduces to (i) $\mu>1/2$, in two dimensions and (ii) $\mu>0$ in three or more dimensions. At $k=0$, the known result that $\lambda(k)-1=O(k^\mu)$ ensures differentiability if $\mu>1$. Since the sum over m in (45) contains oscillatory terms qualitatively similar to the one-dimensional Weierstrass case, we conjecture that if the structure function is differentiated several times, a continuous but nondifferentiable function will result (but we do not investigate this here).

The continuum limit may be calculated as for the Weierstrass random walk, and one finds:

(i) if $0<\mu<2$: $a-1\sim\alpha\Delta$, $b-1\sim\beta\Delta$, $\tau\propto\Delta^\mu$,

$$\frac{\partial}{\partial t}\tilde{P}(\mathbf{q},t) = -\text{const}\cdot|\mathbf{q}|^{\alpha/\beta}\tilde{P}(\mathbf{q},t) \quad (48)$$

(ii) if $\mu=2$: $a-1\sim\alpha\Delta$, $b-1\sim2\alpha\Delta$, $\tau\propto\Delta^2\ln(1/\Delta)$,

$$\frac{\partial}{\partial t}\tilde{P}(\mathbf{q},t) = -\text{const}\cdot|\mathbf{q}|^2\tilde{P}(\mathbf{q},t) \quad (49)$$

The limiting distributions are the isotropic Lévy distribution and the Gaussian distribution, respectively.

5. FRACTAL DIMENSIONS

It has been noted in Section 2 that the Weierstrass walk will generate a hierarchy of clusters of points visited, with about a subclusters per cluster and a linear scaling b between clusters of adjacent order in the hierarchy, for a walk of modest duration. When the duration of the walk increases to infinity, two possibilities arise:

(i) the clusters remain distinguishable for all time;

(ii) the clustering pattern is blurred out by the walker returning to fill in the interstices between clusters.

The latter case arises when the walk is *persistent*, and the former when the walk is *transient*. Persistence for a lattice walk means certainty for return to the origin (and implies certainty of visiting all lattice sites), while for a walk in continuous space it means certainty of return to any neighborhood of the origin (and implies certainty of visiting any neighborhood of any given point). Pólya's theorem[20] establishes the transience of all walks in three or more dimensions, and the persistence of unbiased mean-squared displacement per jump in one or two-dimensions.

It is known[11,21] that persistence of a walk is equivalent to the divergence of $\int [1 - \tilde{p}(\mathbf{q})]^{-1} d^s\mathbf{q}$, the region of integration being any small sphere enclosing $\mathbf{q} = \mathbf{0}$. This leads to the conclusions that (i) the (one-dimensional) Weierstrass random walk is transient if $0 < \mu < 1$ and persistent otherwise,[3] and (ii) the fractal Rayleigh–Pearson walk (in two or more dimensions) is transient if $0 < \mu < 2$.

Having used the term "fractal" somewhat casually in connection with the walks of the present paper we now show how these walks are related to the "fractals" of Mandelbrot[1] and other workers. Mandelbrot has proposed the term "fractal" to denote a set of points having noninteger dimension, when the dimension is inferred from properties of the set according to a particular rule. In the case of a self-similar set, for which a finite subset Ω may be broken into N identical parts, each of which (when magnified by a linear factor λ) is identical with the original, one defines the fractal dimension as

$$D_F = \ln N / \ln \lambda \tag{50}$$

For the random walks considered in the present paper, we have argued that for transient walks a hierarchy of clusters of points visited will be established. This hierarchy is not geometrically self-similar, but rather is self-similar in an average sense, since each cluster is made up of about a subclusters, etc. We therefore interpret

$$\mu = \ln a / \ln b \tag{51}$$

as a fractal dimension, in an average sense, provided that the walk is transient. This is consistent with Mandelbrot's introduction of fractal terminology for random process governed by Lévy (stable) processes.[1] The deduction of the fractal dimension from rigorous analysis of Lévy processes is rather deep, but we have shown here that the choice of a suitable discrete analog leads to a simple understanding of these matters, and we shall enlarge on this in a subsequent publication.

6. RANDOM WALK GENERATING FUNCTIONS

Many important properties of random walks are most easily derived from random-walk generating functions. These functions also appear naturally in continuous time random walk models[26] where a distribution function for pausing times between steps exists. The generating function that corresponds to the walk characterized by (1) is [with $P_0(\mathbf{x}) \equiv \delta(\mathbf{x})$]

$$G(z,\mathbf{x}) = \sum_{n=0}^{\infty} z^n P_n(\mathbf{x}) \tag{52}$$

Then from (2) and (3) with $\tilde{p}(\mathbf{q})$ given by (4)

$$G(z,\mathbf{x}) = \frac{1}{(2\pi)^s} \int \overset{\infty}{\underset{-\infty}{\cdots}} \int \frac{\exp(-i\mathbf{q}\cdot\mathbf{x})\, d^s\mathbf{q}}{1 - z\tilde{p}(\mathbf{q})} \tag{53}$$

When the walks have certain symmetries this integral may be simplified. In a one-dimensional walk on a lattice with lateral spacing Δ, the only possible occupation sites for the walker are at $x = 0, \pm\Delta, \pm 2\Delta, \ldots$. With $P_n(l)$ denoting the probability that $x = l\Delta$ at the nth step, and $G(z,l) \equiv \sum_{n=0}^{\infty} z^n P_n(l)$, it follows from (53) and (21) that

$$G(z,l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\exp(-ikl)}{1 - z\lambda(k)}\, dk \tag{54}$$

In the case of the Rayleigh–Pearson walk, s-dimensional spherical symmetry allows one to reduce the number of integrations in (53). We restrict ourselves here to the cases $s = 2, 3$. Others follow in a similar manner. When $s = 2$, (41) becomes

$$\tilde{p}(\mathbf{q}) = \int_0^{\infty} J_0(|\mathbf{q}|\xi) p_1(\xi)\, d\xi \tag{55a}$$

and when $s = 3$, we use the well-known relationship between $J_{1/2}(x)$ and $\sin x$ to obtain

$$\tilde{p}(\mathbf{q}) = \int_0^{\infty} \frac{\sin(|\mathbf{q}|\xi)}{|\mathbf{q}|\xi}\, p_1(\xi)\, d\xi \tag{55b}$$

In any number of dimensions $\tilde{p}(\mathbf{q})$ for the Rayleigh–Pearson walk depends only on the radial variable $q = |\mathbf{q}|$. Hence the integration in (53) can be performed immediately over the other variables. For example when $s = 2$ and 3 we have, respectively,

$$d^2\mathbf{q} = q\, dq\, d\theta \qquad \text{with} \quad 0 \leqslant \theta < 2\pi \tag{56a}$$

$$d^3\mathbf{q} = q^2 \sin\theta\, d\theta\, d\phi\, dq \qquad \text{with} \quad 0 \leqslant \theta < \pi \quad \text{and} \quad 0 \leqslant \phi < 2\pi \tag{56b}$$

Let $r = |\mathbf{x}|$. In the $s = 2$ case (53) becomes

$$G(z,\mathbf{x}) = \frac{1}{(2\pi)^2}\int_0^\infty \int_0^{2\pi} \frac{q\exp(-iqr\cos\theta)\,d\theta\,dq}{1 - z\tilde{p}(\mathbf{q})}$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{qJ_0(qr)\,dq}{1 - z\tilde{p}(\mathbf{q})} \tag{57}$$

where $\tilde{p}(\mathbf{q})$ is given by (55a). In the $s = 3$ case

$$G(z,\mathbf{x}) = \frac{1}{(2\pi)^3}\int_0^\infty \int_0^{2\pi}\int_0^{\pi} \frac{q^2\sin\theta\exp(-iqr\cos\theta)\,d\theta\,d\phi\,dq}{1 - z\tilde{p}(\mathbf{q})}$$

$$= \frac{1}{\pi^2}\int_0^\infty \frac{q^2\sin qr\,dq}{2qr[1 - z\tilde{p}(\mathbf{q})]} \tag{58}$$

where in this formula $\tilde{p}(\mathbf{q})$ is given by (55b). Note that in general for the Rayleigh–Pearson walk, $G(z,\mathbf{x})$ is a function of $|\mathbf{x}| \equiv r$ alone. In a continuous time random walk composed of an alternating sequence of steps and pauses, with step distribution $p(\mathbf{x})$ and pausing time distribution $\psi(t)$, with $p(\mathbf{x})$ being independent of t, the probability of the walker being at $\mathbf{x}$ at time t is[26]

$$P(\mathbf{x},t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ut}[1 - \psi^*(u)]\,G[\psi^*(u),\mathbf{x}]\,\frac{du}{u} \tag{59}$$

Here $\psi^*(u)$ is the Laplace transform of $\psi(t)$. Properties of this function for various forms of $\psi(t)$ will be discussed elsewhere.

APPENDIX

The small-k behavior of functions defined for $k > 0$ by series of the form

$$\phi(k) = \sum_{n=0}^{\infty} a^{-n} f(b^n k) \tag{A.1}$$

may be formally exhibited by replacing f by its inverse Mellin transform

$$f(b^n k) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (b^n k)^{-p} F(p)\,dp \tag{A.2}$$

interchanging orders of summation and integration, and summing the geometric series $\sum a^{-n} b^{-np}$ to yield a contour integral

$$\phi(k) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} [1 - a^{-1}b^{-p}]^{-1} k^{-p} F(p)\,dp \tag{A.3}$$

A series for $\phi(k)$ at small k is then extracted by translation of the contour and use of the residue theorem. This device has been rigorously implemented by Hardy and Littlewood[23] in the case $f(x) = \exp(-yx)$, where the convergence is sufficiently strong to make a rigorous justification of each step in the analysis easy. To evaluate the structure functions for the Weierstrass and fractal Rayleigh–Pearson walks, the obvious choices of $\cos(x)$ and $\Gamma(\frac{1}{2}s)(\frac{1}{2}x)^{1-x/2}J_{s/2-1}(x)$ for $f(x)$ do not lend themselves easily to a rigorous analysis. However, subtracting 1 from each of these functions enables the integration contour to be placed in a region of superior convergence.

We consider first the fractal Rayleigh–Pearson walk, for which few technical devices are required in the analysis. Taking

$$f(x) = \Gamma(\tfrac{1}{2}s)(\tfrac{1}{2}x)^{1-s/2}J_{s/2-1}(x) - 1 \tag{A.4}$$

we note that[24]

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{-p}2^{p-1}\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}p)}{\Gamma(\frac{1}{2}s-\frac{1}{2}p)}\,dp \tag{A.5}$$

with $-2 < c = \mathrm{Re}(p) < 0$. The integrand has simple poles at $p = 0, -2, -4, \ldots$. Restricting c to the interval $\max(-2, -\mu) < c < 0$, we may justify the interchange of orders of summation and integration by absolute convergence,[19] since

$$|\Gamma(x+iy)| = (2\pi)^{1/2}|y|^{x-1/2}\exp(-\tfrac{1}{2}\pi|y|)\{1+o(1)\} \tag{A.6}$$

as $|y| \to \infty$. We obtain the contour integral

$$\phi(k) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{k^{-p}2^{p-1}\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}p)}{\Gamma(\frac{1}{2}s-\frac{1}{2}p)[1-a^{-1}b^{-p}]}\,dp \tag{A.7}$$

The integrand has poles at $p = 0, -2, -4, \ldots$ and also at $p = p_m = -\mu + 2\pi i m/\ln b$ $(m = 0, \pm 1, \pm 2, \ldots)$, where $\mu = \ln a/\ln b$. We consider the contour integral in Eq. (A.7) taken around the rectangle with corners at $p = c \pm (2M+1)\pi i/\ln b$ and $p = -(2N+1) \pm (2M+1)\pi i/\ln b$, with M, N integral. As $M \to \infty$, the contributions from sides parallel to the real axis vanish and we obtain from the residue theorem the expansion

$$\begin{aligned}\phi(k) = {} & \sum_{n=1}^{N} \frac{(-1)^n\Gamma(\frac{1}{2}s)(\frac{1}{2}k)^{2n}}{n!\,\Gamma(n+\frac{1}{2}s)(1-a^{-1}b^{2n})} \\ & + \frac{(\frac{1}{2}k)^{\mu}}{2\ln b}\sum_{m=-\infty}^{\infty} \frac{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}p_m)}{\Gamma(\frac{1}{2}s+\frac{1}{2}p_m)}\exp\left[-\frac{2m\pi i}{\ln b}\ln\left(\frac{k}{2}\right)\right] \\ & + \frac{1}{2\pi i}\int_{-(2N+1)-i\infty}^{-(2N+1)+i\infty} \frac{k^{-p}2^{p-1}\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}p)}{\Gamma(\frac{1}{2}s-\frac{1}{2}p)[1-a^{-1}b^{-p}]}\,dp\end{aligned} \tag{A.8}$$

In the case when μ is an even integer the series requires minor modification due to the occurrence of a double pole. In particular, if $\mu = 2$ the $n = 1$ and the $m = 0$ terms in the sum are replaced by

$$-\frac{(\frac{1}{2}k)^2\Gamma(\frac{1}{2}s)}{\ln a\Gamma(1+\frac{1}{2}s)}\left[\ln(2/k)+\tfrac{1}{4}\ln a+\tfrac{1}{2}\psi(2)+\tfrac{1}{2}\psi(1+\tfrac{1}{2}s)\right] \qquad \text{(A.9)}$$

where ψ is the usual digamma function $[\psi(z) = (d/dz)\ln\Gamma(z)]$. As is well known, $\psi(z+1) = \psi(z) + z^{-1}$, $\psi(2) = 1 - \gamma$ and $\psi(5/2) = -\gamma - 2\ln 2 + 8/3$, with γ denoting Euler's constant, so that for any value of s, the expression (A.9) is easily evaluated. Equation (45) is established by showing that the contour integral in Eq. (A.8) vanishes as $N \to \infty$ (a trite application of Stirling's formula).

The analysis of the Weierstrass walk proceeds similarly, so long as $\mu > \frac{1}{2}$, with the choice

$$f(x) = \cos(x) - 1 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}\Gamma(p)\cos(\tfrac{1}{2}\pi p)\,dp \qquad \text{(A.10)}$$

taking $\min(-\mu, -2) < c = \mathrm{Re}(p) < -\frac{1}{2}$. When $\mu \leqslant \frac{1}{2}$, it is not possible to secure absolute convergence, but we may circumvent this difficulty by considering instead of Eq. (A.10) a function with a better behaved Mellin transform,[24]

$$f(x) = \cos(x)\exp(-\epsilon x) - 1 \qquad (\epsilon > 0) \qquad \text{(A.11)}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{-p}\Gamma(p)\cos(p\arctan[1/\epsilon])}{(1+\epsilon^2)^{p/2}}\,dp \qquad \text{(A.12)}$$

We find that

$$\sum_{n=0}^{\infty} a^{-n}\exp(-\epsilon k b^n)\cos(b^n k)$$

$$= \sum_{n=0}^{\infty} \frac{(-k)^n(1+\epsilon^2)^{n/2}\cos(n\arctan[1/\epsilon])}{n!\,(1-a^{-1}b^n)}$$

$$+ \frac{k^{\mu}(1+\epsilon^2)^{\mu/2}}{\ln b}\sum_{m=-\infty}^{\infty}\Gamma(p_m)\cos(p_m\arctan[1/\epsilon])$$

$$\times \exp\left[-\frac{2m\pi i}{\ln b}\ln(1+\epsilon^2)^{1/2}\right]\exp\left(-\frac{2m\pi i}{\ln b}\ln k\right) \qquad \text{(A.13)}$$

with appropriate modification if μ is integral, and p_m defined as before. The desired transformation is obtained by taking the limit $\epsilon \to 0$. The uniform convergence of the sum over n allows the limit to be passed through the sum, and so

$$\sum_{n=0}^{\infty} a^{-n}\cos(b^n k) = \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n}}{(2n)!(1-a^{-1}b^{2n})}$$

$$+ \frac{k^{\mu}}{\ln b} \lim_{\epsilon\to 0} \sum_{m=-\infty}^{\infty} \Gamma(p_m)\cos(p_m \arctan[1/\epsilon])$$

$$\times \exp\left[-\frac{2m\pi i}{\ln b}\ln(1+\epsilon^2)^{1/2}\right]\exp\left(-\frac{2m\pi i}{\ln b}\ln k\right) \tag{A.14}$$

We have therefore established Eqs. (32) and (33), with the proviso that the doubly infinite sum of oscillating terms in Eq. (32) may require a convergence factor when $\mu \leqslant \frac{1}{2}$. Elsewhere[3] the authors have derived Eq. (32), using Poisson's summation formula, and given an alternative convergence factor (the series being summed by Abelian means). We remark that a theorem of Kolmogoroff and Seliverstoff[25] ensures the convergence of the oscillatory series in Eq. (32) for almost all values of $\ln k$ when $0 < \mu \leqslant \frac{1}{2}$.

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